Int J Theor Phys (2008) 47: 955–960
DOI 10.1007/s10773-007-9521-0

Phase of a Weakly Interacting Bose System with a Time Spontaneous $U(1)$ Symmetry Breaking

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Received: 11 June 2007 / Accepted: 4 August 2007 / Published online: 18 September 2007

Abstract We have studied the dynamical and geometric phases of a weakly interacting Bose system with a time spontaneous $U(1)$ symmetry breaking by using of the Lewis-Riesenfeld invariant theory. The geometric Aharonov-Anandan phase is also given under the cyclical evolution.

Keywords Phase-Bose system

1 Introduction

Recently, much attention has been paid to the investigation of Bose-Einstein condensation (BEC) in dilute and ultracold gases of neutral alkali-metal atoms using a combination of laser and evaporative cooling [1–3] due to the optical properties [4–9], statistical properties [10–12], phase properties [13, 14], and tunneling effect [15–24].

It is known that the concept of geometric phase was first introduced by Pancharatnam [25] in studying the interference of classical light in distinct states of polarization. Berry [26] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [27]. Samuel et al. [28] generalized the pure state geometric phase further by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. This led Mukunda and Simon [29] to put forward a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [30]. Recently, the geometric phase of the mixed states has also been studied [31–33].

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As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [34] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized in [35] by introducing the concept of basic invariants and used to study the geometric phases in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has developed in some different directions [36–46]. In this letter, by using of the Lewis-Riesenfeld invariant theory, we shall study the dynamical and geometric phases of a weakly interacting Bose system with a time spontaneous $U(1)$ symmetry breaking.

2 Model

For a weakly interacting Bose system, the grand-canonical Hamiltonian can be written by

$$H' = H - \mu N = \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2m} - \mu \right) a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{U_0}{2V} \sum a_{\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_2}^{\dagger} a_{\mathbf{p}_2'} a_{\mathbf{p}_1'} \delta_{\mathbf{p}_1+\mathbf{p}_2, \mathbf{p}_1'+\mathbf{p}_2'}, \tag{1}$$

and the Bogoliubov truncated Hamiltonian with a time spontaneous $U(1)$ symmetry breaking has the following form [47, 48]

$$\begin{aligned} H_\lambda' = {} & -\mu a_0^{\dagger} a_0 + \frac{U_0}{2V} a_0^{\dagger 2} a_0^2 + \sum_{\mathbf{p}\neq 0} \left\{ \left(\frac{1}{2}\varepsilon_{\mathbf{p}} + g_0 a_0^{\dagger} a_0 \right) (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} a_{-\mathbf{p}}) \right. \\ & \left. + \frac{g_0}{2} (a_0^2 a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} + a_0^{\dagger 2} a_{\mathbf{p}} a_{-\mathbf{p}}) \right\} + \sqrt{V} [\lambda(t) a_0 + \lambda^*(t) a_0^{\dagger}], \end{aligned} \tag{2}$$

where $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2m - \mu$, $\lambda(t) = |\lambda(t)| e^{i\eta}$, and $g_0 = U_0/V$. μ is the chemical potential of the system, and $N = \sum_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$ is the total number of the particles. It is apparent that, under the transformation $U = \exp[-i\theta'(a_0^{\dagger} a_0 + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} a_{-\mathbf{p}})]$, the Hamiltonian H_λ' becomes

$$U^{\dagger} H_\lambda' U = H_{\lambda=0}' + \sqrt{V} [\lambda(t) e^{-i\theta'} a_0 + \lambda^*(t) e^{i\theta'} a_0^{\dagger}], \tag{3}$$

we find that H_λ' has the $U(1)$ symmetry, but this symmetry is broken for $\lambda(t) \neq 0$. Introducing the generators of the $SU(1,1)$) group [49, 50]

$$K_0^{\dagger} = \frac{1}{2} a_0^{\dagger 2}, \qquad K_0^{-} = \frac{1}{2} a_0^2, \qquad K_0^3 = \frac{1}{2} \left(a_0^{\dagger} a_0 + \frac{1}{2} \right), \tag{4}$$

and

$$K_{\mathbf{p}}^{\dagger} = a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger}, \qquad K_{\mathbf{p}}^{-} = a_{\mathbf{p}} a_{-\mathbf{p}}, \qquad K_{\mathbf{p}}^3 = \frac{1}{2} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} a_{-\mathbf{p}} + 1), \tag{5}$$

with the commutation relations $[K^3, K^{\pm}] = \pm K^{\pm}$ and $[K^{\dagger}, K^{-}] = -2K^3$. Then we have

$$\begin{aligned} H_\lambda' = {} & -2\mu K_0^3 + 2g K_0^{\dagger} + 2g^* K_0^{-} + \frac{\mu}{2} - 2g_0 |\langle K_0^{\dagger} \rangle|^2 \\ & + \sum_{\mathbf{p}\neq 0} \left(\omega_{\mathbf{p}} K_{\mathbf{p}}^3 + g K_{\mathbf{p}}^{\dagger} + g^* K_{\mathbf{p}}^{-} - \frac{1}{2}\omega_{\mathbf{p}} \right) + \sqrt{V} [\lambda(t) a_0 + \lambda^*(t) a_0^{\dagger}], \end{aligned} \tag{6}$$

here $\omega_{\mathbf{p}} = \varepsilon_{\mathbf{p}} + 2g_0(2\langle K_0^3\rangle - 1/2)$, and $g = U_0/V\langle K_0^-\rangle = |g|e^{-i\varphi'}$. In order to study the phase of a weakly interacting Bose system with a time spontaneous $U(1)$ symmetry breaking, we consider the effective parts of (6), namely,

$$H_\lambda^{eff}(t) = -2\mu K_0^3 + 2gK_0^\dagger + 2g^* K_0^- + \sqrt{V}[\lambda(t)a_0 + \lambda^*(t)a_0^\dagger]. \tag{7}$$

For simplicity, we let $H_\lambda^{eff}(t) = H(t)$ below.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [34]. For a one-dimensional system whose Hamiltonian $H(t)$ is time-dependent, then there exists an operator $I(t)$ called invariant if it satisfies the equation

$$i\frac{\partial I(t)}{\partial t} + [I(t), H(t)] = 0. \tag{8}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$I(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{9}$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = H(t)|\psi(t)\rangle_s. \tag{10}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (10) is different from the eigenfunction $|\lambda_n, t\rangle$ of $I(t)$ only by a phase factor $\exp[i\delta_n(t)]$ for the non-degenerate state, i.e.,

$$|\lambda_n, t_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{11}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (10). Then the general solution of the Schrödinger equation (10) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{12}$$

where

$$\delta_n(t) = \int_0^t dt' \left\langle \lambda_n, t' \middle| i\frac{\partial}{\partial t'} - H(t') \middle| \lambda_n, t' \right\rangle, \tag{13}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$. We can construct the L-R invariant as follows

$$I(t) = y\left\{\frac{1}{2}\sinh\alpha e^{-i\beta}(K_0^\dagger + \sqrt{V}\lambda^* a_0^\dagger) + \frac{1}{2}\sinh\alpha e^{i\beta}(K_0^- + \sqrt{V}\lambda a_0)\right\} + \cosh\alpha K_0^3, \tag{14}$$

where the constant y will be determined below, $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are determined by (8), and satisfy the relations

$$4g^*\cosh\alpha + ye^{i\beta}[(2\mu + \dot\beta)\sinh\alpha - i\dot\alpha\cosh\alpha] = 0, \qquad \sqrt{V}|\lambda|^2 y\sinh\alpha\sin\beta = 0, \tag{15}$$

$$\begin{aligned} &ye^{-i\beta}[i\dot\alpha\sqrt{V}\lambda^*\cosh\alpha + \dot\beta\sqrt{V}\lambda^*\sinh\alpha + i\sqrt{V}\dot\lambda^*\sinh\alpha + (\mu\sqrt{V}\lambda^* - \sqrt{V}\lambda)\sinh\alpha] \\ &\quad + 2y\sqrt{V}\lambda g\sinh\alpha e^{i\beta} + \sqrt{V}\lambda^*\cosh\alpha = 0, \end{aligned} \tag{16}$$

where dot denotes the time derivative.

We can construct the unitary transformation

$$V(t)=\exp\left[\frac{\sigma}{2}K_0^{\dagger}-\frac{\sigma^*}{2}K_0^{-}\right], \tag{17}$$

where $\sigma=\alpha e^{-i\beta}$ and $\sigma^*=\alpha e^{i\beta}$. By making use of the Glauber formulate, one has

$$\begin{aligned} I_V &= V^{\dagger}(t)I(t)V(t)=K_0^3(y\sinh^2\alpha+\cosh^2\alpha)\\ &\quad+\frac{1}{2}(y\sinh\alpha\cosh\alpha+\sinh\alpha\cosh\alpha)(e^{i\beta}K_0^{-}+e^{-i\beta}K_0^{\dagger})\\ &\quad+\left(\lambda^*\sinh\frac{\alpha}{2}+\lambda e^{i\beta}\cosh\frac{\alpha}{2}\right)a_0+\left(\lambda^*e^{-i\beta}\cosh\frac{\alpha}{2}+\lambda\sinh\frac{\alpha}{2}\right)a_0^{\dagger}. \end{aligned} \tag{18}$$

It can be easily seen that when

$$y=-1, \qquad \lambda^*\sinh\frac{\alpha}{2}+\lambda e^{i\beta}\cosh\frac{\alpha}{2}=0, \tag{19}$$

one has $I_V=K_0^3$ which is time-independent. Thus the eigen-equation of the time-independent invariant I_V may be written as

$$I_V|k_0,n_0\rangle=(n_0+k_0)|k_0,n_0\rangle. \tag{20}$$

In terms of the unitary transformation $V(t)$ and the Baker-Campbell-Hausdorff formula [50]

$$V^{\dagger}(t)\frac{\partial V(t)}{\partial t}=\frac{\partial L}{\partial t}+\frac{1}{2!}\left[\frac{\partial L}{\partial t},L\right]+\frac{1}{3!}\left[\left[\frac{\partial L}{\partial t},L\right],L\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial L}{\partial t},L\right],L\right],L\right]+\cdots, \tag{21}$$

where $V(t)=\exp[L(t)]$, one has

$$\begin{aligned} H_V(t) &= V^{\dagger}(t)H(t)V(t)-iV^{\dagger}(t)\frac{\partial V(t)}{\partial t}\\ &=\{-2\mu\cosh\alpha(t)+4|g|\sinh\alpha(t)\cos[\beta(t)-\varphi']\}K_0^3\dot{\beta}(t)[1-\cosh\alpha(t)]K_0^3. \end{aligned} \tag{22}$$

It is easy to find that $H(t)$ differs from I_V only by a time-dependent c-number factor. Thus we can get the general solution of the time-dependent Schrödinger equation (10)

$$|\Psi(t)\rangle_s=\sum_{k_0}\sum_{n_0}C_{k_0,n_0}\exp[i\delta(t)]\hat{V}(t)|k_0,n_0\rangle, \tag{23}$$

with the coefficients $C_{k_0n_0}=\langle k_0,n_0,t=0|\Psi(0)\rangle_s$. The phase $\delta(t)=\delta^d(t)+\delta^g(t)$ includes the dynamical phase

$$\delta^d(t)=(n_0+k_0)\int_{t_0}^{t}\{2\mu\cosh\alpha(t')-4|g|\sinh\alpha(t')\cos[\beta(t')-\varphi']\}dt', \tag{24}$$

and the geometric phase

$$\delta^g(t)=-(n_0+k_0)\int_{t_0}^{t}\dot{\beta}(t')[1-\cosh\alpha(t')]dt'. \tag{25}$$

Particularly, the geometric phase becomes under the cyclical evolution

$$\delta^g(t) = -(n_0 + k_0) \oint [1 - \cosh\alpha(t')] d\beta(t'), \tag{26}$$

which is the known geometric Aharonov-Anandan phase.

4 Conclusions

In this letter, by using of the Lewis-Riesenfeld invariant theory, we have obtained the dynamical and geometric phases of a weakly interacting Bose system with a time spontaneous $U(1)$ symmetry breaking. The geometric Aharonov-Anandan phase is derived under the cyclical evolution.

Acknowledgements This work was partly supported by the Beijing NSF under Grant No. 1072010.

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